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NORMAL AND TANGENTIAL COMPLIANCE FOR CONFORMING BINDER CONTACT I: ELASTIC BINDER

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Abstract—This article engages the studies of normal and tangential conforming contact compliance for a system of two elastic particles bonded by a layer of elastic binder in between. The governing equation of this problem is a Fredholm integral equation of the second kind with singularities of logarithmic type. The exact solution for the unknown interfacial pressure between particle and binder is difficult to obtain. Derivations of compliance are presented in the forms of the upper and lower bounds, and of the best estimate based on physical approximations. The results show that the derived elastic compliances agree favorably with those of 'discretized exact solutions' obtained from numerical methods. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

The subject of layer/binder contact frequently occurs in granular/particulate materials such as asphaltic concrete or cemented sand. This subject is also important in tribology, involving the mechanical behavior of coated materials. Many topics in this area have been investigated in the past years. For example, Muki (1960) studied the problem of contact between a layer and an elastic half space. Goodman and Keer (1975) studied a case of surface layers bounded to a rigid substrate. Bentall and Johnson (1968) worked on a plane strain layered problem which was further studied by Meijers (1968) and Alblas and Kuipers (1970) for both conditions of thin and thick layers. Matthewson (1981) presented a theory of indentation of a soft thin coating by a rigid body. Keer *et al.* (1991) investigated the compliance of coated elastic bodies in contact. Dvorkin *et al.* (1991) employed semi-analytical solutions to examine the normal interaction of two elastic spheres separated by an elastic cementation layer and recently (1994) extended the solutions to examine tangential deformation of two cemented spheres. In addition, many related topics can be found in the books by Johnson (1985) and by Gladwell (1980).

This study is focused on the compliance of a system of two elastic particles bonded together by a thin layer of binder. We aim to derive closed-form relationships between the forces and the relative particle/binder movements in this system. Closed-form expressions are of particular interest because they can be readily employed as a two-particle interaction law and incorporated into discrete element methods for the analysis of a large assembly of particles.

Progression of the article begins with establishing integral equations that govern the interfacial contact pressure distribution between particle and binder. Analyses of the upper and lower bounds with respect to the compliance of the two particle system follow. The best estimate of compliance based on physical approximation is then pursued. The closed-form analytical expressions are compared with the numerical solutions obtained directly from solving the governing integral equations by a discretization technique.

FORMULATION OF THE PROBLEM

We limit our analysis to the condition of two large particles with a thin layer of binder. Under this condition, the problem can be formulated by a set of integral equations (Dvorkin *et al.*, 1994). In this section, we briefly describe the problem and define the variables to be used in the paper. Figure 1 shows an axi-symmetric configuration of two particles bonded by a binder in a cylindrical coordinate. The function $z = h(r)$ represents the geometry of interfacial boundary between the particles and the binder, given by

$$h(r) = h_0\left(1+d\frac{r^2}{a^2}\right) \tag{1}$$

where a is the radius of contact area, h_0 is the thickness of the binder at $r = 0$, and d is the dimensionless shape parameter related to the curvature of particle surface, which is limited in a range $0 \leqslant d < 1$. For a planar surface, d is zero. For a spherical particle, d is given by

$$d = \frac{a^2}{2Rh_0} \tag{2}$$

where R is the radius of the spherical particles.

We denote the constraint modulus E_1 and E_2 Poisson's ratio v_1 and v_2 for the particles and the binder respectively, where the constraint modulus E_1 and E_2 are defined as

$$E_i = \frac{2G_i(1-v_i)}{1-2v_i}; \quad i = 1,2 \tag{3}$$

and G_1 and G_2 are the shear modulus of the particles and the binder respectively.

We intend to derive the normal and tangential compliance of this two particle system with an elastic binder. The governing equations that related force and relative movement of two particles are discussed separately for the normal mode and the tangential mode.

Normal compliance

The relative normal approach δ_z for the two particles is separated into two components: the normal displacement at the binder-particle interface relative to the particle's centroid, $w_1(r)$; and the normal displacement at the binder-particle interface (i.e., at $z = h(r)$) relative to the $z = 0$ plane, $w_2(r)$, given by

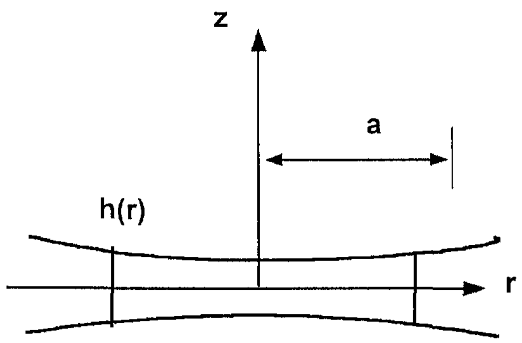

Fig. 1. Sketch of the configuration for a binder-particle system.

$$\delta_z = w_1(r) + w_2(r) \tag{4}$$

Since $z = 0$ is a plane of symmetry, the binder normal displacement vanishes at $z = 0$. For two large particles with a thin layer of binder, we approximate the normal strain to be uniform in the z direction across the binder. The normal displacement $w_2(r)$ contributed from the binder can be expressed as follows:

$$w_2(r) = h(r)\frac{p(r)}{E_2} \tag{5}$$

where $p(r)$ is the interfacial normal pressure between the particles and the binder.

In the analysis of the normal displacement $w_1(r)$ contributed from the particle, due to the assumption of large particle dimension compared to the particle-binder contact area, it is justifiable to pursue the analysis based on a half-space premise. Following the well-known Boussinesq equation, $w_1(r)$ can be related to $p(r)$ by:

$$w_1(r) = \frac{(1-\nu_1^2)}{\pi E_1}\int_0^a\int_0^{2\pi}\frac{p(\rho)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta}{\sqrt{r^2+\rho^2-2r\rho\cos\theta}} \tag{6}$$

Substituting eqns (5) and (6) into eqn (4), we have

$$\delta_z = h(r)\frac{p(r)}{E_2} + \frac{(1-\nu_1^2)}{\pi E_1}\int_0^a \frac{p(\rho)\rho I(\rho, r)\,\mathrm{d}\rho}{\sqrt{r^2+\rho^2}} \tag{7}$$

where $I(\rho, r)$ is defined as

$$I(\rho, r) = I(k) = \int_0^{2\pi}\frac{\mathrm{d}\theta}{\sqrt{1-k\cos\theta}}; \quad k = \frac{2r\rho}{r^2+\rho^2} \tag{8}$$

Integration of the interfacial pressure function, $p(r)$, over the contact area gives the resultant normal contact force P_z

$$P_z = 2\pi\int_0^a p(r)r\,\mathrm{d}r \tag{9}$$

Equations (7) and (9) indirectly provide the compliance relationship between the relative normal approach δ_z and the contact force P_z through the interfacial pressure function.

Tangential compliance

The relative tangential approach in the x-direction δ_x for the two particles is also separated into two components: the tangential displacement at the binder-particle interface relative to the particle's centroid, $u_1(r, \theta)$; and the tangential displacement at the binder-particle interface (i.e., at $z = h(r)$) relative to the $z = 0$ plane, $u_2(r, \theta)$, given by

$$\delta_x = u_1(r, \theta) + u_2(r, \theta) \tag{10}$$

The tangential displacement vanishes at the plane of symmetry $z = 0$. Since the particle is assumed to be large compared to the particle-binder contact area and the binder is a thin layer, we approximate the tangential strain to be uniform in the z direction across the binder, and the following relation can then be derived:

$$u_2(r,\theta) = h(r)\frac{q(r,\theta)}{G_2} \tag{11}$$

where G_2 is the shear modulus of binder, $q(r,\theta)$ is the interfacial tangential pressure between the particles and the binder.

Similarly we pursue the analysis of $u_1(r,\theta)$ based on the half-space premise (Johnson, 1985). Thus the tangential displacement δ_x for the two contact bodies is now equal to the summation of $u_1(r,\theta)$ and $u_2(r,\theta)$ with an error of the order $(v_1)^2$ (Dvorkin *et al.*, 1994)

$$\delta_x = h(r)\frac{q(r,\theta)}{G_2} + \frac{1}{2\pi G_1}\int_0^a\int_0^{2\pi} q(\rho,\phi)F(r,\rho,\theta,\phi,v_1)\rho\,\mathrm{d}\phi\,\mathrm{d}\rho \tag{12}$$

where

$$F(r,\rho,\theta,\phi,v_1) = \left\{\frac{1-v_1}{\xi} + v_1\frac{(r\cos\theta-\rho\cos\phi)^2}{\xi^3}\right\}$$

$$\xi^2 = (r\cos\theta-\rho\cos\phi)^2 + (r\sin\theta-\rho\sin\phi)^2 \tag{13}$$

where G_1 and v_1 are respectively the shear modulus and Poisson's ratio of the particle. Integration of the interfacial pressure function, $q(r,\theta)$, over the contact area gives the resultant tangential force P_x

$$P_x = \int_0^a\int_0^{2\pi} q(r,\theta)r\,\mathrm{d}\theta\,\mathrm{d}r \tag{14}$$

Again the governing eqns (12) and (14) provide the compliance relationship between the relative tangential approach δ_x and the tangential force P_x through the unknown interfacial pressure function, $q(r,\theta)$.

In fact, the interfacial pressure functions, $q(r,\theta)$ or $p(r)$, can be determined by simultaneously solving eqns (12) and (14) or eqns (7) and (9). Unfortunately, the governing eqns (7) and (12) are Fredholm integral equations of the second kind with kernels which have logarithmic singularities. For this type of integral equations, closed-form analytical solutions are difficult to obtain. However, they can be solved numerically, for example, using the quadrature technique (Dvorkin *et al.*, 1991) or the technique presented in Appendix A.

ANALYTICAL SOLUTIONS FOR TWO EXTREME CASES

The analytical solutions of the interfacial pressures $p(r)$ in eqn (7) and $q(r,\theta)$ in eqn (12) are known for two extreme cases, namely (1) rigid particle case (i.e., $E_1 \to \infty$ and $G_1 \to \infty$ while E_2 and G_2 are finite), and (2) rigid binder case (i.e., E_1 and G_2 are finite while $E_2 \to \infty$ and $G_1 \to \infty$). The compliance relationships under these two extreme conditions are described in this section.

Rigid particle case

In the rigid particle case, the relative movement of the two contact bodies is contributed only from the deformation of binder. Thus

$$\delta_z = h(r)\frac{p(r)}{E_2} \tag{15}$$

$$\delta_x = h(r)\frac{q(r,\theta)}{G_2} \tag{16}$$

Subsequently, for the rigid particle case, the corresponding normal interfacial pressure, denoted as $p_1(r)$, is given by

$$p_1(r) = \frac{P_z h_0}{\pi a^2 h(r) X} \tag{17}$$

and the interfacial tangential pressure $q(r, \theta)$ becomes independent of the variable θ (denoted as $q_1(r)$) and reads

$$p_1(r) = \frac{P_x h_0}{\pi a^2 h(r) X} \tag{18}$$

where

$$X = \frac{\ln(1+d)}{d} \tag{19}$$

and d is the shape parameter defined in eqn (1).

Thus the normal and tangential compliance relationships between the contact force P_z and the relative approach δ_z become

$$\delta_x = C_{1z} P_z ; \quad C_{1z} = \frac{h_0}{\pi a^2 E_2 X} \tag{20}$$

$$\delta_x = C_{1x} P_x ; \quad C_{1x} = \frac{h_0}{\pi a^2 G_2 X} \tag{21}$$

Since the particles are rigid, C_{1x} and C_{1z} represent the binder compliances.

Rigid binder case

In the rigid binder case, which represents the well known rigid punch problem, the normal interfacial pressure, denoted as $p_2(r)$, is given by

$$p_2(r) = \frac{P_z}{2\pi a^3}(a^2 - r^2)^{-1/2} \tag{22}$$

and the tangential pressure, $q(r, \theta)$, is again dependent only on r (denoted as $q_2(r)$):

$$q_2(r) = \frac{P_x}{2\pi a}(a^2 - r^2)^{-1/2} \tag{23}$$

For this case, the normal and tangential compliances are:

$$\delta_z = C_{2z} P_z ; \quad C_{2z} = \frac{1 - \nu_1^2}{2aE_1} \tag{24}$$

$$\delta_x = C_{2x} P_x ; \quad C_{2x} = \frac{2 - \nu_1}{8aG_1} \tag{25}$$

Since the binder is rigid, C_{2x} and C_{2z} represent the particle compliances.

UPPER BOUND SOLUTION

Explicit compliance relationships are easily derived for the two extreme conditions. However, for general conditions, analytical solutions to eqn (7) and eqn (12) are difficult to obtain. Therefore, in what follows, we seek the approximate solutions which represent the upper and lower bounds, and the best estimated solution based on the physical approximations.

In order to find an upper bound solution for the integral eqn (7), we multiply $r/h(r)$ by eqn (7) and then integrate the equation over the range $0 \leqslant r \leqslant a$, which yields:

$$\delta_z = C_{1z}P_z + 4C_{2z}\frac{h_0}{a\pi X}\int_0^a f(\rho)p(\rho)\rho\,\mathrm{d}\rho \tag{26}$$

where C_{1z} and C_{2z} are the compliances of the two extreme cases given previously in eqns (20) and (24), and the function $f(\rho)$ is:

$$f(\rho) = \int_0^a \frac{I(\rho, r)r\,\mathrm{d}r}{h(r)\sqrt{(r^2+\rho^2)}} \tag{27}$$

Similarly, for the tangential compliance, we multiply $r/h(r)$ by eqn (12), then integrate the equation with respect to the variables (r, θ) over the range $0 < r < a$, $0 < \theta < 2\pi$. The following expression is obtained:

$$\delta_x = C_{1x}P_x + \frac{4C_{2x}h_0}{a\pi^2 X(2-v_1)}\int_0^{2\pi}\int_0^a f_x(\rho, \phi, v_1)q(\rho, \phi)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \tag{28}$$

where C_{1x} and C_{2x} are the compliances of the two extreme cases given previously in eqns (21) and (25), and

$$f_x(\rho, \phi, v_1) = \int_0^{2\pi}\int_0^a F(\rho, r, \theta, \phi, v_1)r\,\mathrm{d}r\,\mathrm{d}\theta \tag{29}$$

We now have converted the original set of governing eqns (7) and (12) into a new set of governing eqns (26) and (28) which are in terms of the compliances of the two extreme conditions. This conversion does not make it easier for analytical evaluation because eqns (26) and (28) still contain integrals of the unknown interfacial pressure functions. However, this conversion has transformed the governing equations to be suitable for the use of *Chebyshev's inequality for integrals*. The principles of *Chebyshev's inequality* can greatly simplify the integrals containing the pressure function and thus allow us to obtain simple solutions to this problem.

It can be proved that $f(\rho)$ in eqn (26) is a monotonically decreasing function since its derivative with respect to ρ is negative in the range of $0 \leqslant \rho \leqslant a$ (see Appendix B). From eqns (17) and (22), it can be easily verified that the function $p(\rho)\rho$ in eqn (26) is a monotonically increasing function under both the extreme conditions (i.e. rigid binder case and rigid particle case). Indeed, the function $p(\rho)\rho$ is monotonically increasing for any pairs of E_1, E_2 and G_1, G_2 as verified from the numerical method given in Appendix A.

Applying *Chebyshev's inequality* to the integral in eqn (26) (see Appendix C):

$$\int_0^a f(\rho)p(\rho)\rho\,\mathrm{d}\rho \leqslant \frac{1}{a}\int_0^a f(\rho)\,\mathrm{d}\rho\int_0^a p(\rho)\rho\,\mathrm{d}\rho \leqslant \frac{aP_z}{2\pi h_0}H(d)$$

$$H(d) = 5.699 - 2.404d + 1.495d^2 - 1.079d^3 + 0.841d^4 - 0.689\frac{d^5}{1+d} \tag{30}$$

Substituting the inequality of eqn (30) into (28), the upper bound solution for the normal compliance is derived as

$$\delta_z \leqslant (C_{1z}+b_1 C_{2z})P_z \tag{31}$$

where

$$b_1 = \frac{2H(d)}{\pi^2 X} \tag{32}$$

For tangential compliance, it is easily seen that

$$f_x(\rho,\phi,v_1) \leqslant f_x(\rho,\phi,0) = 2\pi f(\rho) \tag{33}$$

where $f(\rho)$ is defined in eqn (27). We can use both eqns (28) and (33) to derive:

$$\delta_x \leqslant C_{1x}P_x + \frac{8C_{2x}h_0}{a\pi X(2-v_1)}\int_0^{2\pi}\int_0^a f(\rho)q(\rho,\phi)\rho\,\mathrm{d}\rho \tag{34}$$

Again, the function $q(\rho,\phi)\rho$ is monotonically increasing with respect to ρ. The application of *Chebyshev's inequality* to the integral in the right hand side of eqn (28) leads to the upper bound solution for the tangential compliance:

$$\delta_x \leqslant \left(C_{1x}+C_{2x}\frac{2b_1}{2-v_1}\right)P_x \tag{35}$$

where b_1 is defined in eqn (32).

LOWER BOUND SOLUTION

For the lower bound analysis, we convert the original set of governing equations into a new set of governing equations using a different set of multipliers. We multiply $p_2(r)r$ by eqn (7), and integrate the equation over the range $0 \leqslant r \leqslant a$. We multiply $q_2(r)r$ by eqn (12), and integrate the equation over the range $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$. Thus:

$$\delta_z = C_{1z}\frac{2\pi^2 a^2 X}{h_0 P_z}\int_0^a p(r)rp_2(r)h(r)\,\mathrm{d}r + C_{2z}P_z \tag{36}$$

$$\delta_x = C_{1x}\frac{\pi a^2 X}{h_0 P_x}\int_0^{2\pi}\int_0^a q(r,\theta)rq_2(r)h(r)\,\mathrm{d}r + C_{2x}P_z \tag{37}$$

The new set of governing equations are also in terms of compliances of extreme conditions. From the definitions of $h(r)$, $p_2(r)$ and $q_2(r)$ (eqns (1, 22 and 23)), it is easily seen that $p_2(r)h(r)$ and $q_2(r)h(r)$ are both monotonically increasing functions in the range $0 \leqslant r \leqslant a$. Since $p(r)r$ is a monotonically decreasing function, the application of the *Chebyshev's inequality* results in:

$$\int_0^a p(r)rp_2(r)h(r)\,\mathrm{d}r \geqslant \frac{1}{a}\int_0^a p(r)r\,\mathrm{d}r\int_0^a p_2(r)h(r)\,\mathrm{d}r = 0.25(1+0.5d)h_0 P_z\frac{P_z}{2\pi a^2} \tag{38}$$

Thus the lower bound solution for the normal compliance is derived as

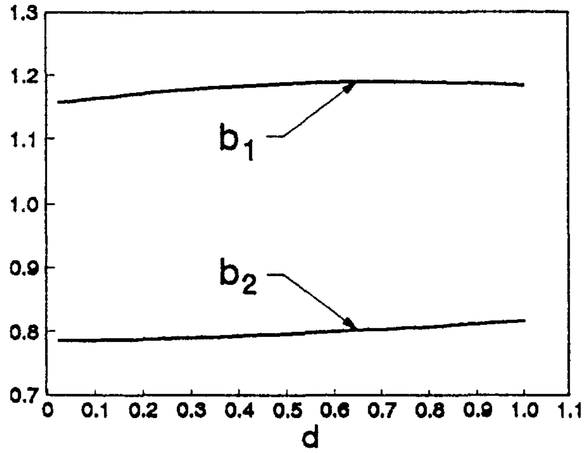

Fig. 2. The values for b_1 and b_2 vs d.

$$\delta_z \geqslant (b_2 C_{1z} + C_{2z}) P_z \tag{39}$$

where

$$b_2 = \frac{\pi}{4}(1+0.5d)X < 1 \tag{40}$$

Similarly, applying *Chebyshev's inequality* to eqn (37) results in the lower bound solution for the tangential compliance:

$$\delta_x \geqslant (b_2 C_{1x} + C_{2x}) P_x \tag{41}$$

Based on the upper and lower bound solutions, the true normal compliance must be between $(C_{1z}+b_1C_{2z})$ and $(b_2C_{1z}+C_{2z})$, and the true tangential compliance must be between $(C_{1x}+b_1C_{2x})$ and $(b_2C_{1x}+C_{2x})$. It is useful to examine the range of values of b_1 and b_2 which represents a measure of the range of upper and lower compliance bounds. For this purpose, the values of b_1 in eqn (21) and b_2 in eqn (25) vs the shape parameter, d, are plotted in Fig. 2. It is noted from Fig. 2 that the extreme value is about 1.2 for b_1 and 0.8 for b_2. Consequently, the maximum relative difference between the upper and lower bounds of compliance is from 18 to 20%.

BEST ESTIMATE BASED ON PHYSICAL APPROXIMATION

In this section, we seek the best estimated compliance relationship. Two estimates are conducted: we pursue the first estimate based on the set of governing eqns (26) and (28); and the second estimate based on the set of governing eqns (36) and (37). Instead of using the principles of *Chebyshev's inequality* to estimate the integral containing the unknown interfacial pressure function, we now select a suitable form for the pressure function to obtain the best estimated solutions for the governing equations.

For eqn (26), we select the interfacial pressure $p_2(\rho)$ given in eqn (22) for the rigid punch problem as the substituting pressure function. It can be seen that when C_{1z} is negligible (i.e., rigid binder case), eqn (16) corresponds to a rigid punch problem. This substitution yields the exact expression of a rigid punch problem. On the other hand, when C_{1z} becomes dominant and C_{2z} is negligible (i.e., the rigid particle case), the contribution of the integral is trivial to the solution of eqn (26), thus the form of $p(\rho)$ makes little difference. Therefore, substituting the $p(\rho)$ with $p_2(\rho)$ is a physically consistent choice, and it leads to the following simple compliance relationship:

$$\delta_z = (C_{1z}+C_{2z})P_z \tag{42}$$

When eqn (36) serves as the starting point of the second estimate, we employ the similar argument in the first estimate and thus select the interfacial pressure function $p_1(\rho)$ given in eqn (17) for the rigid particle case as the substituting pressure function. When C_{2z} is negligible (rigid particle case), the substitution yields an exact solution. When C_{2z} becomes dominant (rigid binder case), the contribution of the integral is trivial in eqn (36), thus the form of $p(\rho)$ makes little difference to the compliance. Therefore, the second estimate is obtained by substituting $p(\rho)$ in eqn (36) with $p_1(\rho)$, and it yields, surprisingly, the same relationship as the one in the first estimate (i.e., eqn (42)).

Similarly, for tangential compliance, we select the rigid binder pressure $q_2(\rho)$ to substitute the unknown pressure distribution $q(\rho, \phi)$ in the integral of eqn (28) and the rigid particle pressure $q_1(\rho)$ to substitute the unknown pressure distribution $q(\rho, \phi)$ in the integral of eqn (37). Both processes lead to the identical result:

$$\delta_x = (C_{1x}+C_{2x})P_x \tag{43}$$

Equations (42) and (43) are therefore considered the best estimated compliance relationships. They satisfy the two extreme cases: (1) rigid particle case ($E_1 \to \infty$ and E_2 finite); and (2) rigid binder case (E_1 finite and $E_2 \to \infty$). In addition, the best estimated compliances fall in between the upper and lower bounds, i.e., the following inequalities are always satisfied:

$$b_2 C_1 + C_2 < C_1 + C_2 < C_1 + b_1 C_2 \tag{44}$$

$$b_2 C_{1x} + C_{2x} < C_{1x} + C_{2x} < C_{1x} + \frac{2b_1}{2-v_1} C_{2x} \tag{45}$$

Equations (42) and (43) indicate that the best estimated compliance can be obtained by selecting $b_1 = 1$ and $b_2 = 1$. The best estimated overall compliance corresponds to a serial connection of the two compliances C_1 and C_2 as schematically shown in Fig. 3, where C_1 represents the compliance of particle and C_2 represents the compliance of binder.

We now compare the best estimated analytical compliances with the compliances numerically calculated using the method described in Appendix A. Whereby we introduce a normalized dimensionless compliance C_{norm} which is defined as follows:

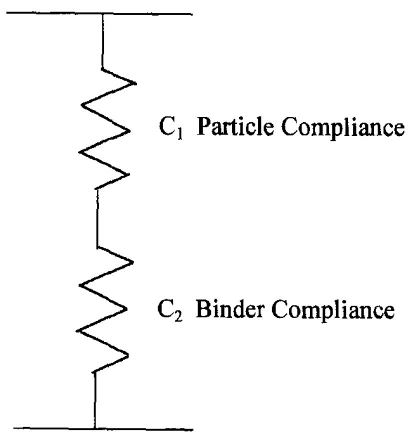

Fig. 3. Schematical representation for the compliance of an elastic particle-binder system.

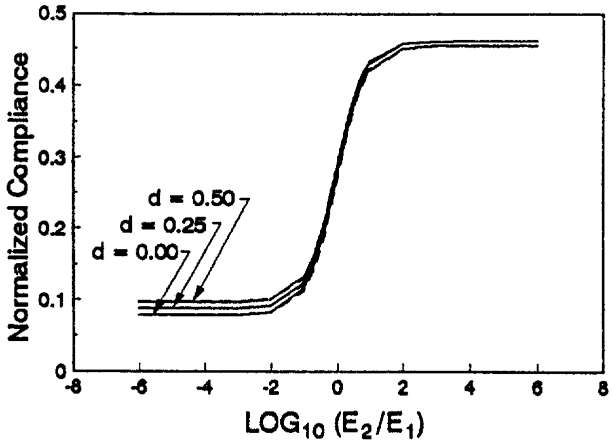

Fig. 4. A comparison of normalized compliances (analytical results are in solid lines; numerical results are in dashed lines. $h_0 = 0.25a$).

$$C_{norm} = \frac{C_{1z}+C_{2z}}{C_{ref}} \quad \text{(analytical)}$$

$$C_{norm} = \frac{C_{num}}{C_{ref}} \quad \text{(numerical)} \tag{46}$$

where C_{1z} and C_{2z} are the compliances given in eqns (20) and (24), C_{num} is the numerically calculated compliance defined in eqn (A13). C_{ref} is the reference compliance defined by:

$$C_{ref} = \frac{1}{a}\left(\frac{1}{E_1}+\frac{1}{E_2}\right) \tag{47}$$

where E_1 and E_2 are the constraint modulus respectively for the particle and binder and a is the contact area between particle and binder.

The comparisons of both numerical and analytical C_{norm} are shown in Fig. 4. The parameters are: $h_0 = 0.25a$; $v_1 = 0.2$; with three different values of d; $d = 0$, $d = 0.25$ and $d = 0.5$. The analytical results agree very well with the numerically calculated results that the differences between curves are almost indiscernible. It is noted that the agreement between analytical and numerical solutions is also found for the tangential compliance.

CONCLUSIONS

For the compliance of an elastic particle-binder system, the governing equation is a Fredholm integral equation of the second kind with singularities of logarithmic type. The exact solutions for such integrals are difficult to obtain. Even solved numerically, special care needs to be taken for the convergency problems associated with the singularities. This paper makes a use of the principles of *Chebyshev's inequality*. This approach allows simplifications of the governing equations and yields remarkably simple closed-form expressions for the upper and lower bound solutions of compliances.

The derived upper bound compliance is in the form of $(C_1+b_1C_2)$ where C_1 represents the compliance of a deformable particle-rigid binder system (given in eqns 20 and 21), C_2 represents the compliance of a rigid particle-deformable binder system (given in eqns 24 and 25), and b_1 is a constant greater than one (eqn 32). The derived lower bound compliance is in the form of $(b_2C_1+C_2)$ where b_2 is a constant less than one (eqn 40). The best estimated compliance is in the form of (C_1+C_2) corresponding to $b_1 = b_2 = 1$.

The best estimates of compliances have been found to be favorably in agreement with the numerical solutions and thus can be considered as good approximate solutions for this type of problem. The closed-form relationship of the particle-binder system can be used as the particle interaction law in a discrete element analysis for the deformation analysis of a large assembly of particles.

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APPENDIX A: NUMERICAL DISCRETIZATION ALGORITHM

A useful result in this study is the following derived inequality:

$$ZJ(k) < I(k) < J(k) \tag{A1}$$

where

$$Z = \frac{\pi}{2\ln\dfrac{\sqrt{2}+1}{\sqrt{2}-1}} = 0.891107 \tag{A2}$$

$I(k)$ is defined in eqn (8); $J(k) = J_1(k)+J_2(k)$; and

$$J_1(k) = \frac{4}{\sqrt{1+k}}\ln\left[\frac{\sqrt{2}+\sqrt{1+k}}{1-k}\right]; \quad J_2(k) = 4\ln\left[1+\frac{\sqrt{2}}{\sqrt{1+k}}\right] \tag{A3}$$

Based on the bounds of $I(k)$ in eqn (A1), we approximate $I(k)$ as the weighted average of the two bounds; $I(k) = I_a(k)$, $0 \leqslant k \leqslant 1$; where

$$I_a(k) = (Z(1-k)+k)J_1(k)+ZJ_2(k) \tag{A4}$$

Note that $I_a(k) = I(k)$ at $k = 0$ and $k \to 1$. The comparison between $I(k)$ and $Ia(k)$ shown in Fig. A1 indicates that $I_a(k)$ is a good approximation of $I(k)$.

We then replace $I(k)$ in eqn (7) with $I_a(k)$, and have:

$$\delta_z = \frac{h(r)p(r)}{E_2} + \lambda\int_0^a \frac{p(\rho)}{\sqrt{r^2+\rho^2}} I_a(\rho, r)\rho \, d\rho \tag{A5}$$

$$\lambda = \frac{(1-\nu^2)}{\pi E_1} \tag{A6}$$

Equation (A5) is solved using a numerical discretization technique. The segment $0 \leqslant \rho$, $r \leqslant a$ is divided into N intervals. The unknown function $p(r)$ is linearly interpolated in each interval. Gaussion $2M$ point interpolation is

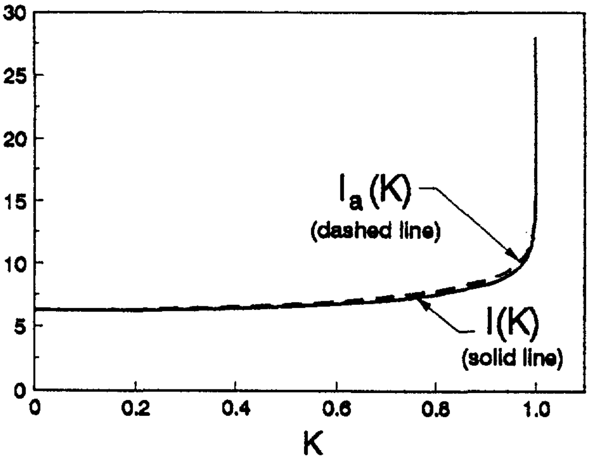

Fig. A1. A comparison of $I(k)$ and $I_a(k)$.

employed in numerical integration over each interval. Multiple values of N and M are selected to assure stability of the solution. The discretized format for eqn (A5) in its indicia notation now reads

$$\delta_z e_i = L_{ij} f_j \quad i,j = 1,2,\ldots,N+1 \tag{A7}$$

where

$$e_j = 1; \quad f_j = f(r_j) \tag{A8}$$

and the coefficient tensor L_{ij} is given by

$$L_{ij} = \frac{h(r_j)\delta_{ij}}{E_2} + \lambda\Delta \sum_{m=1}^{2M} \frac{I_a(\bar{\rho}_j + \Delta\xi_m, r_i)\eta_m}{\sqrt{r_i^2 + (\bar{\rho}_j + \Delta\xi_m)^2}}; \quad (i,j = 1,2,\ldots,N+1) \tag{A9}$$

where ξ_m and η_m are Gaussian interpolation points and coefficients respectively. δ_{ij} is the Kronecker delta, and

$$\bar{\rho}_i = 0.5(\rho_i + \rho_{i+1}); \quad \Delta = \frac{1}{2N}; \quad r_j = (j-1)\frac{a}{N} \tag{A10}$$

Thus the discretized $p(r)$ can be obtained from inversion of eqn (A7), it yields:

$$p_j = p(r_j) = \delta_z L_{ij}^{-1} e_i; \quad i = 1,2,\ldots,N+1 \tag{A11}$$

and P_z in eqn (10) can be expressed numerically:

$$P_z = 4\pi\Delta L_{ij}^{-1} e_j r_i \delta_z \tag{A12}$$

Consequently, a numerical compliance C_{num} can be introduced:

$$C_{num} = \frac{\delta_z}{P_z} = [4\pi\Delta L_{ij}^{-1} e_j r_i]^{-1} \tag{A13}$$

APPENDIX B: MONOTONICALLY DECREASING PROPERTY FOR $f(\rho)$

In the coordinate system (s, ϕ) given in Fig. A2, an equivalent form of $f(\rho)$ in eqn (18) can be expressed by:

$$f(\rho/a) = \frac{2a}{bh_0} \int_0^{\pi} tg^{-1}\left[\frac{X_1}{X_2}\right] \frac{d\phi}{X_2} \tag{B1}$$

where X_1 and X_2 are defined as:

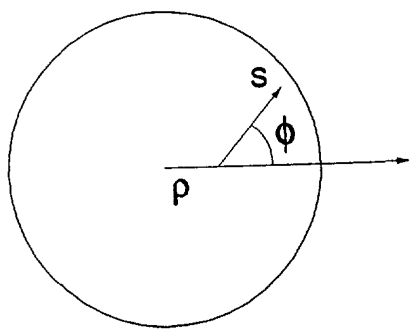

Fig. A2. The (s, ϕ) coordinate system.

$$X_1 = \sqrt{a^2-\rho^2\sin^2\phi}; \quad X_2 = \sqrt{\rho^2\sin^2\phi+\frac{a^2}{d}} \tag{B2}$$

It can be easily shown that the derivative of the integrand in r.h.s. of eqn (B1) with respect to ρ is

$$\frac{\mathrm{d}f(\rho/a)}{\mathrm{d}\rho} = \frac{2a}{bh_0}\int_0^\pi tg^{-1}\left[\frac{X_1}{X_2}\right]\frac{X_1'}{X_2}\mathrm{d}\phi + \frac{2a}{bh_0}\int_0^\pi \frac{1}{X_2}\frac{\mathrm{d}\phi}{1+\left(\frac{X_1}{X_2}\right)^2}\left(\frac{X_1'X_2-X_1X_2'}{X_2^2}\right) \tag{B3}$$

where

$$X_1' = -\rho\sin^2\phi < 0; \quad X_2' = \frac{\rho\sin^2\phi}{X_2} > 0 \tag{B4}$$

Therefore it can be seen that the derivative of $f(\rho)$ is negative in $0 \leqslant \rho \leqslant a$ and $f(\rho)$ is a monotonically decreasing function. The following inequality holds true:

$$\frac{1}{1+d}f(\rho=1, d=0) \leqslant f(\rho) \leqslant f(\rho=0) \tag{B5}$$

$$\frac{4a}{h_0(1+d)} \leqslant f(\rho) \leqslant \frac{2a\pi}{h_0\sqrt{d}}tg^{-1}(\sqrt{d}) \tag{B6}$$

APPENDIX C: CHEBYSHEV'S INEQUALITY FOR INTEGRALS

Let $f(x), g(x)$ be non-negative integrable functions on $[a, b]$, and $f(x)$ is monotonic increasing, $g(x)$ is monotonic decreasing; then Chebyshev's inequality for integrals is applicable to the following integral and it yields:

$$(b-a)\int_a^b f(x)[g(a)-g(x)]\,\mathrm{d}x \geqslant \int_a^b f(x)\,\mathrm{d}x\int_a^b [g(a)-g(x)]\,\mathrm{d}x. \tag{C1}$$

An equivalent expression for eqn (C1) is

$$(b-a)\int_a^b f(x)g(x)\,\mathrm{d}x \leqslant \int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)\,\mathrm{d}x. \tag{C2}$$